7.1 Laplace Transform

the Laplace transform of f(t) is defined as

e^{-st} is the kernel

$$\boxed{\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt}$$

this is one example of integral transforms (another commonly used one is the Fourier transform)

let's look at the transforms of simple functions

$$\mathcal{L}\{1\} = F(s) = \int_0^\infty 1 \cdot e^{-st}\, dt$$

treat s as a "constant"

$$= \lim_{a\to\infty} \int_0^a e^{-st}\, dt$$

t is variable

$$= \lim_{a\to\infty} -\frac{1}{s} e^{-st} \Big|_{t=0}^{t=a} = \lim_{a\to\infty} \left(-\frac{1}{s} \underbrace{e^{-sa}} + \frac{1}{s} \right)$$

must go to 0 for integral to converge

$s > 0$

$$\boxed{\mathcal{L}\{1\} = \frac{1}{s}, \quad s>0}$$

$$\mathcal{L}\{t\} = \int_0^\infty t e^{-st}\,dt = \lim_{a\to\infty}\int_0^a t e^{-st}\,dt$$

by parts

$u = t \qquad dv = e^{-st}dt$

$du = dt \qquad v = -\frac{1}{s}e^{-st}$

$uv - \int v\,du$

$$= \lim_{a\to\infty}\left(-\frac{t}{s}e^{-st}\Big|_0^a + \int_0^a \frac{1}{s}e^{-st}\,dt\right)$$

$$= \lim_{a\to\infty}\left(-\frac{t}{s}e^{-st}\Big|_0^a - \frac{1}{s^2}e^{-st}\Big|_0^a\right)$$

$$= \lim_{a\to\infty}\left(-\frac{a}{s}e^{-sa} - \frac{1}{s^2}e^{-sa} + \frac{1}{s^2}\right)$$

must go to 0

so, $s>0$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}, \quad s>0}$$

similarly, we can show that $\mathcal{L}\{t^2\} = \frac{2}{s^3}, \quad s>0$

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s>0$$

$$\mathcal{L}\{e^{ct}\} = \int_0^\infty e^{ct} e^{-st}\,dt$$

$$= \lim_{a\to\infty} \int_0^a e^{(c-s)t}\,dt$$

$$= \lim_{a\to\infty} \frac{1}{c-s} e^{(c-s)t} \Big|_0^a = \lim_{a\to\infty} \left(\frac{1}{c-s} \underbrace{e^{(c-s)a}}_{\text{must go to } 0} - \frac{1}{c-s} \right)$$

$c-s<0$, so $s>c$

$$\boxed{\mathcal{L}\{e^{ct}\} = \frac{1}{s-c}, \quad s>c}$$

Laplace transform is linear (because integration is linear)

$$\mathcal{L}\{f(t)+g(t)\} = \int_0^\infty [f(t)+g(t)]\,e^{-st}\,dt$$

$$= \int_0^\infty [f(t)e^{-st} + g(t)e^{-st}]\,dt$$

$$= \int_0^\infty f(t)e^{-st}\,dt + \int_0^\infty g(t)e^{-st}\,dt$$

$$= \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}$$

$$\mathcal{L}\{\underset{\text{constant}}{c}\cdot f(t)\} = \int_0^\infty c\cdot f(t)e^{-st}\,dt$$

$$= c\cdot\int_0^\infty f(t)e^{-st}\,dt = c\cdot\mathcal{L}\{f(t)\}$$

$$\mathcal{L}\{3+5e^{-2t}-10t^3\}$$

$$= \mathcal{L}\{3\}+\mathcal{L}\{5e^{-2t}\}-\mathcal{L}\{10t^3\}$$

$$= 3\,\mathcal{L}\{1\}+5\,\mathcal{L}\{e^{-2t}\}-10\,\mathcal{L}\{t^3\}$$

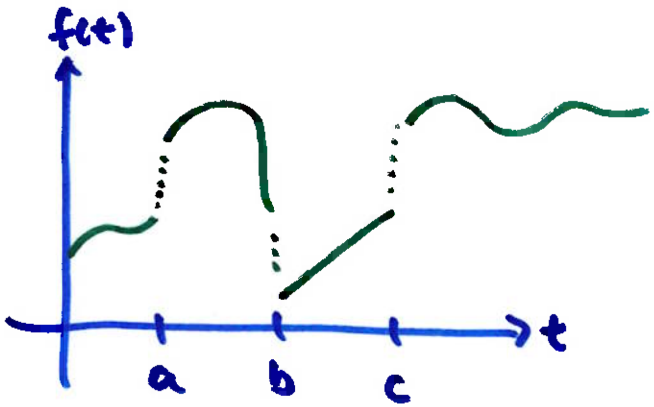
$$= 3\cdot\frac{1}{s}+5\cdot\frac{1}{s+2}-10\cdot\frac{3!}{s^4}\quad,\ s>0$$

Do all functions $f(t)$ have Laplace transforms?

$$\mathcal{L}\{f(t)\} = \int_0^\infty f(t)e^{-st}\,dt$$ $\rightarrow$ whether we can do the integral

$\rightarrow$ as long as the function is <u>piecewise continuous</u>

(finite number of discontinuities)



$$\int_0^\infty f(t)e^{-st}\,dt$$

$$= \int_0^a \cdots + \int_a^b \cdots + \int_b^c \cdots + \int_c^\infty \cdots$$

for example, $f(t) = \begin{cases} 2t+1 & 0 \le t < 3 \\ e^t & 3 \le t < \infty \end{cases}$

$$F(s) = \int_0^3 (2t+1)e^{-st}\,dt + \int_3^\infty e^t e^{-st}\,dt$$

$\vdots$

$$= \frac{1}{s} - \frac{7e^{-3s}}{s} + \frac{2}{s^2} - \frac{2e^{-3s}}{s^2} + \frac{e^{-3(s-1)}}{s-1}, \quad s > 1$$

Laplace transform: $\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)\, e^{-st}\, dt$

Inverse Laplace transform: $\mathcal{L}^{-1}\{F(s)\} = f(t)$

$$= \underbrace{\frac{1}{2\pi i} \lim_{T \to \infty} \int_{\gamma - iT}^{\gamma + iT} e^{st} F(s)\, ds}_{\text{usually not used in practice}}$$

usually, we do table look up for both transforms